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Optimization of the Hausdorff distance between convex polyhedrons in R^3 ⋆

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Abstract: In this article, approximation of sets is under consideration using convex polyhedrons in the three dimensional Euclidean space. In the problem statement, it is necessary to find such disposition of two given polyhedrons A and B that the Hausdorff distance between them obtains the minimal value. Elements of convex analysis, non-smooth analysis, and numerical geometry are used for construction numerical algorithms solving this problem. Numerical algorithms are implemented in the software whose efficiency is demonstrated in applications.

Keywords: Hausdorff distance, Hausdorff metric, polyhedron, optimization, subdifferential, convex set, Chebyshev center.

1. INTRODUCTION

This article is devoted to approximation of sets using convex polyhedrons in Euclidean space $\mathbf{R}^3$. The considered problem arises in the optimal control theory and the differential games theory when it is necessary to construct different types of feasible sets Krasovskii and Subbotin [1974]. For instance, such types of feasible sets can be presented by attainability sets or sets of positional absorption in differential games. Let us note that ellipsoidal methods and algorithms for approximation of attainability sets can be successfully applied in a wide range of tasks Kurzhanski and Valyi [1997]. Commonly, these sets can be presented as non-convex polyhedrons in the three-dimensional Euclidean space.

Let us consider a non-convex polyhedron and assume that it is possible to split it into several convex polyhedrons with arbitrary number (principally, large number) of vertices. Then the problem arises to approximate these convex polyhedrons using more simple polyhedrons with less number of vertices, e.g. parallelepipeds. This problem passes to another one: it is necessary to find such disposition of two given polyhedrons that the Hausdorff distance Hausdorff [1957] between them obtains the minimal value. It should be mentioned that this problem arises also in the pattern recognition theory Koutroumbas and Theodoridis [1999]. For solving this problem, we effectively apply elements of convex analysis Rockafellar [1997] and non-smooth analysis Aubin and Ekeland [1984] in conjunction with computational geometry Preparata and Shamos [19888] and realize them in the elaborated numerical algorithms and in the software complex. We provide particular applications which demonstrate efficiency of the proposed methodology.

While working with sets arising in control problems and differential games it is often necessary to construct set approximations using simplex structures and to estimate how close these approximations are located to the original set. Closedness of sets in the sense of the Hausdorff metric is a key criteria for selection of a substitution for the original set.

We examine sets in the three dimensional Euclidean space $\mathbf{R}^3$. Let us consider two arbitrary sets $A, B \subset \mathbf{R}^3$. We pose the problem: to find relative disposition for these sets which provides the minimal Hausdorff metric value between them

$$d(A, B) = \max\{h(A, B), h(B, A)\}.$$

Here

$$h(A, B) = \max_{a \in A} \min_{b \in B} \|a - b\| \tag{1}$$

is the Hausdorff deviation from A to B.

Let us freeze the location of the set A and shift the set B by its parallel transition to the location $B + \{x\} = \{b + x\colon b \in B\}$ along vector x, $x \in \mathbf{R}^3$. So, every vector $x \in \mathbf{R}^3$ defines the location $B + \{x\}$. One can uniquely determine the Hausdorff distance between sets A and $B + \{x\}$, and define the characteristic function $F(x) = d(A, B + \{x\})$ via formula (1). Thus, the problem under consideration transforms to finding minimum for the characteristic function $F(x)$, $x \in \mathbf{R}^3$. Let us note that authors studied similar problems for planar sets in Lakhtin and Ushakov [2005], Lakhtin, Lebedev, and Ushakov [2014]. For elaboration of corresponding minimization algorithms one can use methods for calculation of the Hausdorff distance presented in Alt, Braß, Godau, Knauer, and Wenk [2003].

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2. PROPERTIES OF THE CHARACTERISTIC FUNCTION $F(X)$

Let consider the case when sets A and B are convex polyhedrons. For each convex set $Y \subset \mathbf{R}^3$ the distance function $\rho(x,Y) = \min_{y\in Y} \|x-y\|$ between set Y and point $x \in \mathbf{R}^3$ is a convex function. From convex analysis it is known (e.g. Rockafellar [1997]) that the maximum value of a convex function on a convex polyhedral set is attained at vertices. Thus, in (1) one can consider only vertices a_i and b_j instead of all points of polyhedrons, i.e.

$$a_i \in A, \quad i = 1,\dots,N_a; \quad b_j \in B, \quad j = 1,\dots,N_b;$$

$$d(A,B) = \max\Big\{ \max_{i=1,\dots,N_a} \rho(a_i,B), \max_{j=1,\dots,N_b} \rho(b_j,A) \Big\}. \quad (2)$$

Here numbers N_a and N_b denote the amount of vertices in sets A and B, respectively. Here and further symbols a_i and b_j stand for vertices of sets A and B.

It is known Sukharev, Timokhov, and Fedorov [1986] that if $f_1,\dots,f_m$ are convex functions defined on an open convex set X then the subdifferential of the maximum function $f(x) = \max_{i=1,\dots,m} f_i(x)$ has the following structure

$$\partial f(x) = \mathrm{co}\Big(\bigcup_{i\in I(x)} \partial f_i(x) \Big), \quad x \in X,$$

where index $I(x) = \{i: f_i(x) = f(x)\}$ indicates the involved functions, and symbol $\mathrm{co}(\cdot)$ denotes the convex hull.

Definition 1. Projection $\pi(a,B)$ of point a on a convex compact set B is a closest point from set B to point a in the sense of the Euclidean metric.

Let us note that if a set is non-convex then one can obtain more than one projection but in the case of convex sets the projection is always unique Rockafellar [1997].

In our case of convex sets A and B, we can write down the formula defining the characteristic function $F(x)$ as follows

$$F(x) = \max\Big\{ \max\{f_i(x): i = 1,\dots,N_a\},$$
$$\max\{f_j(x): j = 1,\dots,N_b\}\Big\}, \quad \text{where}$$
$$f_i(x) = \rho(a_i, B + \{x\}) \text{ if } i = 1,\dots,N_a,$$
$$f_j(x) = \rho(b_j + x, A) \text{ if } j = 1,\dots,N_b.$$

Basing on this representation, one can obtain the formula for the subdifferential of the characteristic function $F(x)$ in the the following way. The set X coincide in this case with the whole space $\mathbf{R}^3$, and the index $I(x)$ is given by relations

$$I(x) = I_A(x) \cup J_B(x),$$
$$I_A(x) = \big\{i: \rho(a_i, B + \{x\}) = F(x)\big\},$$
$$J_B(x) = \big\{j: \rho(b_j + x, A) = F(x)\big\}.$$

The index $I_A(x)$ is determined by the set of numbers for those vertices of the set A whose distance to the polyhedron $B+\{x\}$ equals to the Hausdorff distance between sets A and B. The index $J_B(x)$ of numbers for vertices of the polyhedron $B+\{x\}$ is defined in the analogous way. Thus, the subdifferential of the characteristic function $F(x)$ looks as follows $\partial F(x) = \mathrm{co}\,\{L_A(x) \cup L_B(x)\}$,

$$L_A(x) = \Big\{ -\frac{a_i - \pi(a_i, B + \{x\})}{\|a_i - \pi(a_i, B + \{x\})\|}: \exists i \in I_A(x) \Big\},$$

$$L_B(x) = \Big\{ \frac{(b_i + x) - \pi((b_i + x), A)}{\|(b_i + x) - \pi((b_i + x), A)\|}: \exists j \in J_B(x) \Big\}.$$

The set $L_A(x)$, in the case of its nonemptiness, is presented by the set of unit vectors directed from vertices of the polyhedron A with numbers from the index $I_A(x)$ towards the polyhedron $B+\{x\}$. Analogously, the set $L_B(x)$, in the case of its nonemptiness, is the set of unit vectors aimed at the set A from vertices of the polyhedron $B+\{x\}$ with numbers from $J_B(x)$.

Let us note that the subdifferential $\partial F(x)$ for each point $x \in \mathbf{R}^3$ is a nonempty set which can be defined by no more than $N_a + N_b$ vectors. Geometrically, the subdifferential $\partial F(x)$ is the convex hull of such unit vectors which indicates directions from the polyhedron A to the polyhedron $B+\{x\}$ at which the Hausdorff distance between sets A and $B+\{x\}$ is achieved.

Since the characteristic function $F(x)$ is a convex continuous function with bounded Lebesgue sets then it has point of minimum x^*. Necessary and sufficient conditions of minimum for the characteristic function $F(x)$ are formulated in terms of the subdifferential of this function in the following way (see, for example, Sukharev, Timokhov, and Fedorov [1986], Rockafellar [1997]):
point x^* is point of minimum of the characteristic function $F(x)$ if and only if $\mathbf{0} \in \partial F(x^*)$, $\mathbf{0} = (0,0,0) \in \mathbf{R}^3$. In the general case, point of minimum is not unique.

3. SUBGRADIENT METHOD

For finding minimum of the characteristic convex function $F(x)$, which arises in the case of convex polyhedrons A and B, we apply numerical subgradient methods. One approach which can be used for such minimum search of a nonsmooth convex function belongs to N.Z. Shor Shor [1979] and can be presented by the following iterative algorithm:

$$x^{k+1} = x^k - \gamma_k \frac{h_k}{\|h_k\|}, \quad h_k \in \partial F(x^k). \quad (3)$$

According to this formula, at each iteration the shift of the size γ_k is implemented in the direction opposite to the subgradient. Weakness of this method consists in impossibility of determine the step size γ_k without additional information and also in necessity of selection of some arbitrary subgradient from the subdifferential. In this context, there exist different algorithms for the step size correction and subgradient selection.

For Shor's method a series of step size values should satisfy two conditions. First, it should be a sequence converging to zero. Second, the corresponding sum of the series should be divergent. Taking these two properties into account, we use the sequence $\gamma_k = \gamma_0/k$ for implementation of the algorithm. It is also reasonable to take the Hausdorff distance between sets A and B as an initial value for the step size $\gamma_0 = d(A,B)$.

It should be mentioned that strictly speaking the algorithm (3) can not be labeled as a descent method where sequence $F(x^k)$ is monotonically decreasing. Nevertheless, it is shown Polyak [1983], Shor [1979] one can speak about convergence in terms of the best value of the function $F(x)$ achieved at the current moment. Let us note that since the series $\sum_{k=1}^{\infty} \gamma_k$ is a divergent one then it is quite clear that the algorithm (3) can not converge quickly. Besides, there exist difficulties with obtaining the precision estimate.

In our research we develop several computer versions of the algorithm (3) of the minimum search for the characteristic function $F(x)$. Let us note that in practical realization of these algorithms we deal very often with the situation when there exists exactly one point on one or another polyhedron A, B, the Euclidean distance from which to another polyhedron equals to the Hausdorff distance $d(A, B + \{x\})$. This situation happens since the geometric locus of points x for which the index $I(x)$ consists of two or more elements is the combination of a finite number of two-dimensional, one-dimensional and zero-dimensional manifolds embedded in the three-dimensional space. For example, if we consider two similar cubes as sets A and B then such geometric locuses are presented by combination of three planes passing through the center of cube A in parallel to its faces. Therefore, in many cases it is necessary to select for formula (3) the unique vector h_k from the subdifferential $\partial F(x^k)$.

4. COMPOSITE METHOD

We develop the composite method which allows to improve the rate of convergence and accuracy estimates in comparison with the subgradient method. This new composite method is based on constructions proposed in Lebedev and Ushakov [2012], Lebedev, Uspenskii and, Ushakov [2014], Lakhtin, Lebedev, and Ushakov [2014]. Algorithm of Chebyshev center calculation for polyhedron with large quantity of vertexes is presented in Lebedev and Ushakov [2015].

Definition 2. Garkavi [1962, 1964] The Chebyshev center of a compact set $M \subset \mathbf{R}^3$ is such point $c(M)$, that satisfies the following conditions

$$h(M, \{c(M)\}) = \inf\{h(M, \{x\}) \colon x \in \mathbf{R}^3\}. \tag{4}$$

The value (4) is called the Chebyshev radius $r(M)$ of a compact set $M \subset \mathbf{R}^3$.

The next scheme constitutes the basis of the developed iterative algorithm:

$$x^{k+1} = x^k + c\big(W(x^k)\big), \tag{5}$$

Here the map $W(x)$ is defined by the relation

$$W(x) = \Big\{(a_i - \pi(a, B + \{x\})) \colon i = 1, \ldots, N_a\Big\}$$
$$\cup \Big\{ -(b_j + x - \pi((b_j + x), A)) \colon j = 1, \ldots, N_b\Big\}.$$

This method provides shift of the point x in the direction where minimum of the characteristic function $F(x)$ is achieved.

Theorem 1. Let the characteristic function $F(x)$ reaches its minimum at the point x^k. Then the iterative formula (5) provides stabilization of the algorithm sequence $x^{k+1} = x^k$.

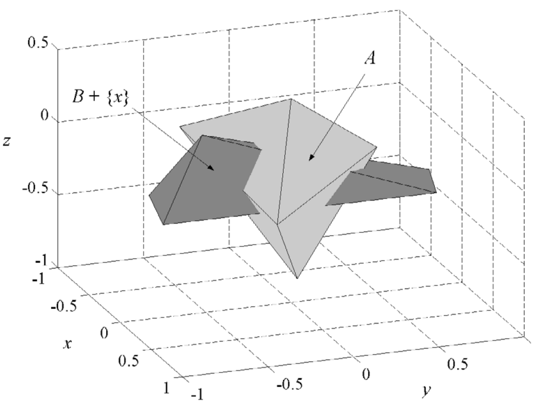

Fig. 1. Polyhedrons A and $B + \{x\}$ in Example 1: Chart 1.

Proof. Let introduce notation for the minimal value of the characteristic function, $F(x^k) = \mu$. Consider the set determined by the formula

$$G(x^k) = \{a_i - \pi(a_i, B + \{x\}) \colon i \in I_A(x)\}$$
$$\cup \{-(b_j - \pi(b_j + x, A)) \colon j \in J_B(x)\}\,.$$

As one can see, the following relation is valid, $\mu \partial F(x^k) = \operatorname{co} G(x^k)$. Since the Hausdorff distance is non-negative by definition, then $\mu \geq 0$. Hence, sets $\partial F(x^k)$ and $\operatorname{co} G(x^k)$ are homothetical with the homothetic center at the origin. If x^k is a point of a local minimum of the characteristic function $F(x)$, then according to the necessary condition of optimality we have $\mathbf{0} \in \partial F(x^k)$. Hence, inclusion $\mu\mathbf{0} = \mathbf{0} \in \operatorname{co} G(x^k)$ takes place.

Properties of the Chebyshev center in the Euclidean space imply that the origin can be located in the convex hull of a finite set vectors with equal norms if and only if the origin itself is the Chebyshev center for this set Garkavi [1964, 1962]. Vectors $a_i - \pi(a_i, B + \{x\})$, $i \in I_A(x)$, and $b_j - \pi(b_j + x, A)$, $j \in J_B(x)$, due to their construction have norms equal to the value μ. Hence, we get $c\big(G(x^k)\big) = \mathbf{0}$, and $r\big(G(x^k)\big) = \mu$.

Let us consider now the Chebyshev center and the radius of the set $W(x^k)$. According to construction for all vectors $w_i \in W(x)$ the estimate $\|w_i\| \leq d(A, B + \{x^k\}) = \mu$ takes place, and, thus, the following inequality is valid, $r(W) \leq \mu$. On the other hand, we have the relation $G(x^k) \subseteq W(x^k)$, which implies the inequality $r(W) \geq mu$. Thus, we get the relation $r(W) = \mu$, which means $c(W(x^k)) = \mathbf{0}$, due to the fact that for any point $w \neq \mathbf{0}$ from the uniqueness of the Chebyshev center it follows that $h(W(x^k), w) \geq h(G(x^k), w) > h(G(x^k), \mathbf{0}) = \mu$. Finally, according to formula (5) we obtain the identical value of the new iteration $x^{k+1} = x^k + c(W(x^k)) = x^k + \mathbf{0} = x^k$, and the algorithm sequence is stabilized.

Theorem 1 means that the composite method is stable in the case when it gets into position providing the minimum value of the Hausdorff distance between polyhedrons.

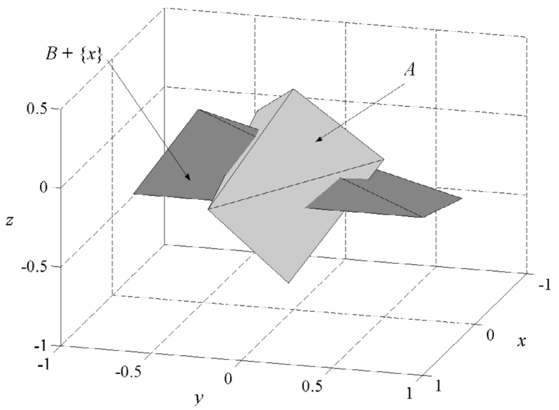

Fig. 2. Polyhedrons A and $B+\{x\}$ Example 1: Chart 2.

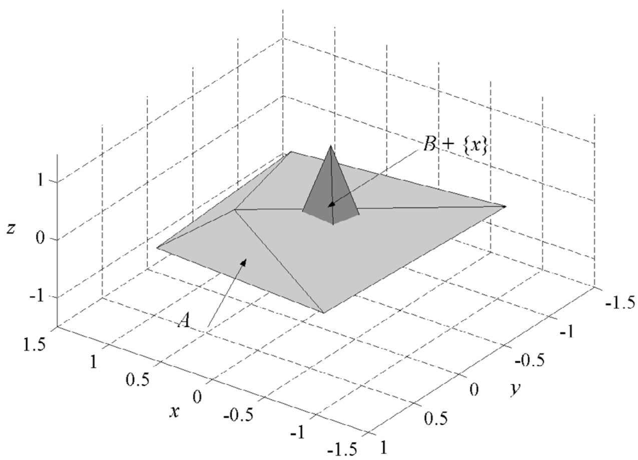

Fig. 3. Polyhedrons A and $B+\{x\}$ in Example 2: Chart 1.

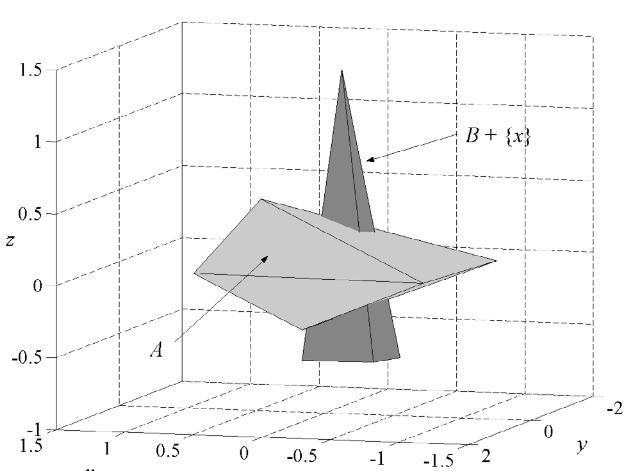

Fig. 4. Polyhedrons A and $B+\{x\}$ in Example 2: Chart 2.

5. APPLICATIONS OF THE COMPOSITE METHOD

The software complex was developed by authors basing on algorithms of the composite method. The application package MATLAB Kwon [1997] is used for implementation of the software complex .

Further, we consider several examples where we apply for solution the composite method

Example 1. Let set A be the convex polyhedron with vertices

$$\{a_i\}_{i=1}^5 = \{(0,0,0.4), (-0.7,-0.4,0), (0.6,-0.3,-0.2), (0,0.5,0), (0.1,0,-0.8)\},$$

and B be the convex polyhedron with vertices

$$\{b_i\}_{i=1}^5 = \{(0.3,-0.5,0.5), (0.5,0.8,0), (0.5,-0.8,0), (0,-0.7,0), (0.1,0.9,0)\}.$$

It is necessary to find vector x such the the Hausdorff distance between A and $B+\{x\}$ reaches its minimal value.

Calculations according algorithms of the composite method provide the value $x = (-0.2929, -0.0215, -0.2872)$ for the minimum point of the Hausdorff distance with the precision estimate $\Delta x = 10^{-3}$. The Hausdorff distance between the given polyhedrons is $d(A, B+\{x\}) = 0.5128$. Besides, the composite method allows to get this result in 5 iterations. The disposition of polyhedrons A and B is shown on Fig. 1 and Fig. 2.

We realized also the subgradient method and made comparison of results. The subgradient method produces the following value for the minimum point of the Hausdorff distance $\widetilde{x} = (-0.2938, -0.0218, -0.2868)$ and reaches it in 500 iterations. In these calculations the Hausdorff distance is evaluated at the level $d(A, B+\{\widetilde{x}\}) = 0.5132$, that is larger than the value provided by the composite method.

Example 2. Let set A be the convex polyhedron with vertices

$$\{a_i\}_{i=1}^6 = \{(0.4,0.3,0.5), (0.8,0.7,0), (-0.9,0.8,0), (-1,-1.1,0), (0.9,-0.9,0), (0.1,0.3,-0.4)\},$$

and set B be the convex polyhedron with vertices

$$\{b_i\}_{i=1}^5 = \{(0.1,-0.1,2), (-0.2,0.2,0), (-0.3,-0.2,0), (0.2,-0.3,0), (0.3,0.3,0)\}.$$

It is necessary to find vector x which provides minimum for the Hausdorff distance $d(A, B+\{x\}$.

The coordinates of the minimum point for the Hausdorff distance are estimated by the composite method at the level $x = (-0.1544, -0.1368, -0.6215)$. The minimal Hausdorff distance between polyhedrons A and $B+\{x\}$ is given by the value $d(A, B+\{x\}) = 1.025$. The composite method allows to get this result in 5 iterations as in the previous example. The disposition of polyhedrons A and B is presented on Fig. 3 and Fig. 4. The subgradient method realizes the minimum point in 500 steps but its result in the value of the Hausdorff distance is again a little bit worse than the result of the composite method: $\widetilde{x} = (-0.1557, -0.1362, -0.6222)$, and $d(A, B+\{\widetilde{x}\}) = 1.0266$.

Results of calculations in both examples illustrate efficiency and high precision of the developed software complex based on algorithms of the composite method. The visualization block of the software provides an opportunity to observe positions of polyhedrons in the three dimensional Euclidean space.

6. CONCLUSION

The paper is devoted to solution of the minimization problem for the Hausdorff distance between two convex compact polyhedrons. An iterative algorithm called the composite method is elaborated for finding minimum point of the characteristic function describing the Hausdorff distance. The stability property is proved for the composite method which provides stabilization of an iterative sequence at the minimum point. Examples are given which illustrate efficiency and high precision of the proposed composite method.

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